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RESEARCH REPORT No. 170-5

Equivalent Circuits for Narrow Dielectric Blocks in Waveguides

by

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This Report was Prepared and Written under
Contract No. W28-099-ac-170
with
Army Air Forces, Air Materiel Command
Watson Laboratories
Red Bank, N. J.
January, 1948

New York University
Washington Square College
Mathematics Research Group
Research Report No. 170-5

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Title page
19 numbered pages

January, 1948

CONTENTS

Article	Page
Abstract	1
1. Introduction	2
2. The Equivalent Circuit	3
3. The Integral Equation	5
4. The Even Case	6
5. The Odd Case	9
6. Variational Formulation of the Impedance Elements	11
7. Fields in an Infinite Block	13
8. The Finite Block Results	15
9. Conclusion	18

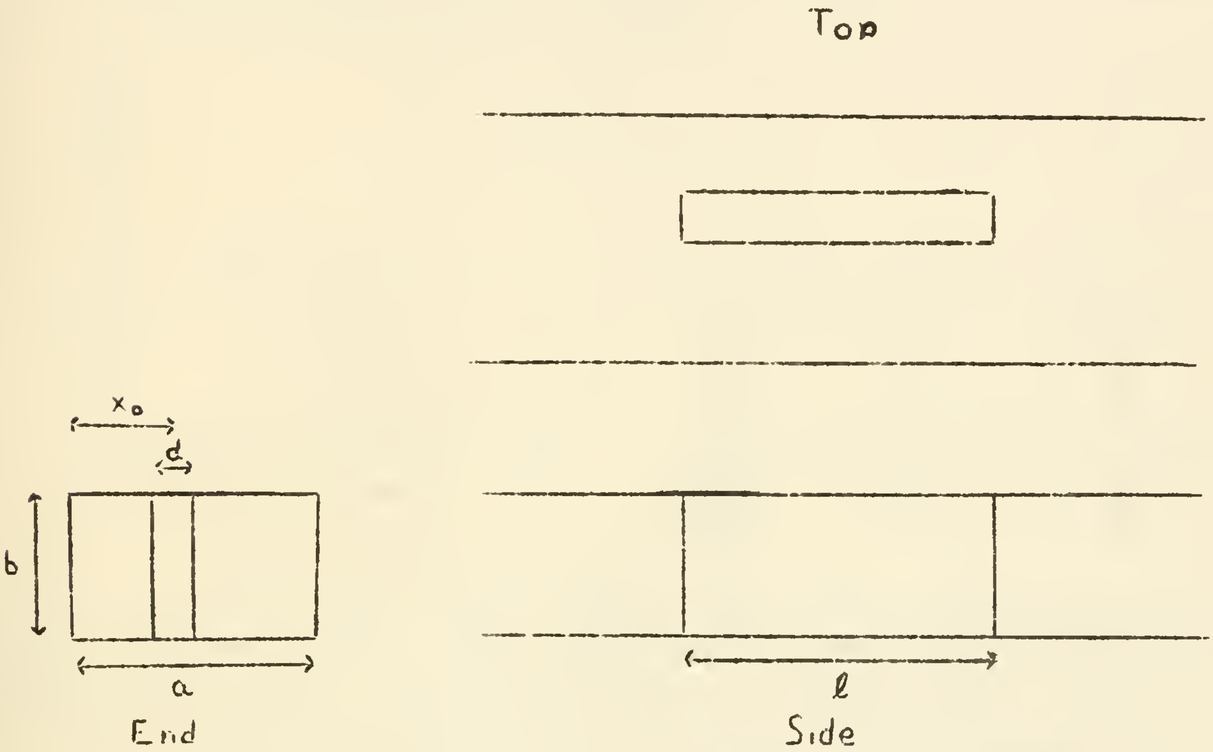
ABSTRACT

This paper investigates the transmission properties of a rectangular waveguide in which there is inserted a parallel rectangular dielectric strip of finite length. The equivalent circuit is determined by variational methods due to Julian Schwinger. Formulas are given for the standing wave ratio due to the dielectric block. These are shown to be quite accurate when the block is narrow compared to the width of the guide. Graphs are appended showing the dependence of the standing wave ratio on the various parameters of the problem.

1. Introduction

The equivalent circuit for a circularly cylindrical dielectric post in a rectangular waveguide has been determined by Julian Schwinger. He gives results for the circuit elements when the post radius is not too large and the dielectric constant is not too great. It is the purpose of this report to find the equivalent circuit when a rectangular dielectric block is placed in a waveguide.

We shall consider only dielectric blocks which are placed parallel to the short side of the guide and extend completely across. Thus the geometrical situation is as follows:



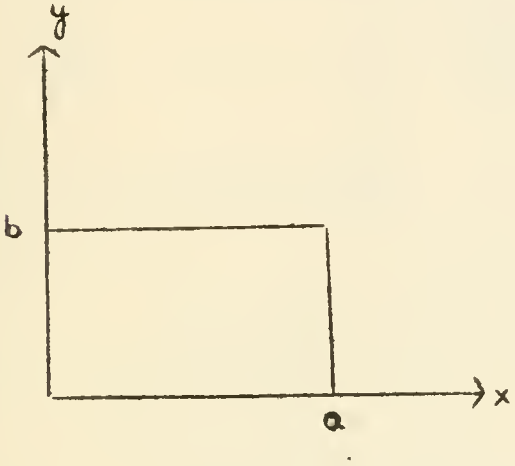
The methods whereby a discontinuity in a waveguide may be represented by an equivalent circuit were developed by Julian Schwinger in a series of lectures given at the M.I.T. Radiation Laboratories. Notes on these lectures have been prepared and distributed but, since the material is not too widely known, portions of it will be developed here. An alternative method of procedure, due to Whinnery and Jameson, (P.I.R.E. Feb. 1944, pp. 98-114), proved too difficult to apply in the problem in question.

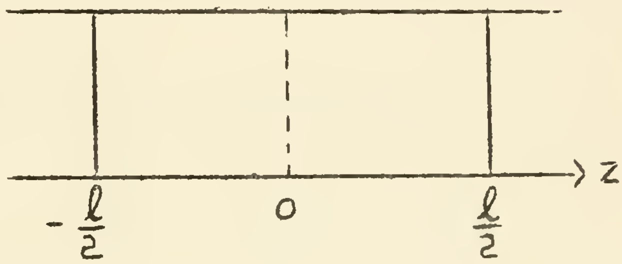
Suppose that the dimensions of the waveguide are such that only the lowest mode, i.e., TE_{01} mode, can propagate. Then, regardless of the nature of the discontinuity, the fields at great distances will be given solely by lowest mode

terms. The effect of the discontinuity on the fields at great distances may accordingly be lumped into a description of the lowest mode fields. Voltages and currents may be defined, and the whole waveguide problem reduced to a transmission line problem, with the discontinuity represented by an equivalent circuit.

2. The Equivalent Circuit

Let us use a coordinate system in which the z-axis is along the guide, with $z = 0$ at the center of the block, and the x and y axes are parallel respectively to the long and short transverse dimensions of the guide. The origin is at the lower left hand corner of the guide.

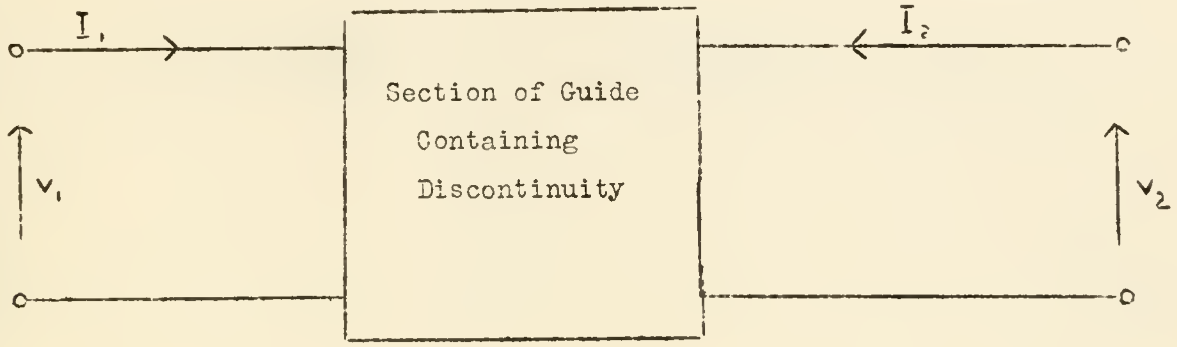




The incident field consists solely of lowest mode waves, which have only a y-component electric field. Because of the nature of the obstacle no other components of electric field will be generated, since all boundary conditions can be satisfied without them; and all field components will be independent of y, since the incident field is. Therefore let us write $E_y = \varphi(x,z)$. The magnetic field may be derived from E_y as $H_x = -\frac{1}{i\omega\mu}\frac{\partial\varphi}{\partial z}$; $H_z = \frac{1}{i\omega\mu}\frac{\partial\varphi}{\partial x}$. The time factor $e^{-i\omega t}$ is understood throughout, and we will eventually replace i by -j, in accordance with engineering practice.

The dielectric block may be somewhat lossy. Under these circumstances, the dielectric constant may be considered to be complex, and the analysis does not change in form. In the end we replace the complex dielectric constant by its real and imaginary parts.

We may conveniently define voltages and currents to obtain the equivalent circuit. The voltage will be expected to be a measure of the lowest mode electric field, and the current a measure of the lowest mode transverse magnetic field. These voltages and currents will then obey transmission line equations, since the field components of the lowest mode fields obey them. By superposition, these voltages and currents will be linearly related. The sense of the voltages and currents will be as in the figure below.



The relations between voltages and currents will be:

1) $V_1 = Z_{11} I_1 + Z_{12} I_2$

$V_2 = Z_{21} I_1 + Z_{22} I_2$

The reciprocity theorem yields $Z_{12} = Z_{21}$, and the symmetry of the obstacle around the $z = 0$ plane yields $Z_{11} = Z_{22}$. Hence the relations between voltages and currents are determined by the two parameters Z_{11} and Z_{12}. To find these, will be the goal of this paper.

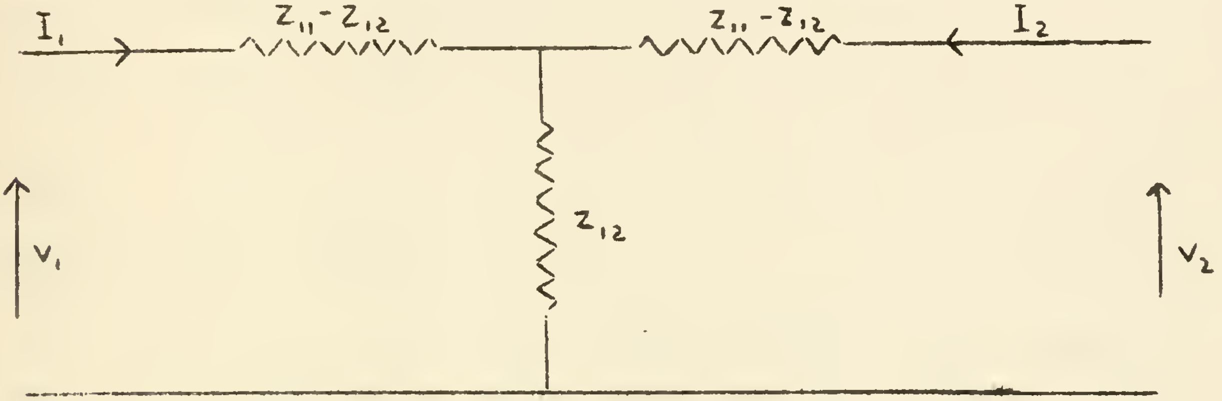
We can represent the total electric field by a sum of even and odd functions. In the even case we have $\varphi_e(x,z) = \varphi_e(x,-z)$, so $V_1 = V_2 = V_e$. From Equation 1) it follows that $I_1 = I_2 = I_e$, so there is a discontinuity in the current at $z = 0$, from the assumed positive directions of I_1 and I_2. This corresponds to an open circuit at $z = 0$. From Equation 1) it follows that:

2) $$\frac{V_e}{I_e} = Z_{11} + Z_{12}$$

In the odd case, we have $\varphi_o(x,z) = -\varphi_o(x,-z)$, and therefore $V_1 = -V_2 = V_o$. From Equation 1) we get $I_1 = -I_2 = I_o$. Hence the current is continuous at $z = 0$, while the voltage is discontinuous, and we have a pure series situation. This corresponds to short-circuiting the guide at $z = 0$. We have from Equation 1)

$$3) \quad \frac{V_o}{I_o} = Z_{11} - Z_{12}$$

The circuit representing the discontinuity is:



3. The Integral Equation

In order to obtain the circuit elements, it is necessary to have expressions for the fields. The analysis for a dielectric block is formally the same as it is for a dielectric post. The latter has been developed by Schwinger, and we shall transcribe parts of it here.

The most convenient way to obtain the fields is by means of a Green's function. Thus, let us define a function $G(x,z,x',z')$ as such that $i\omega\mu G$ is the y-component electric field at a point x, z, produced by a unit current filament in the y direction at x', z'. Let $j_\rho(x,z)$ be the polarization current in the block. Then we obtain the equation:

$$4) \quad \varphi(x,z) = \varphi^{in}(x,z) + i\omega\mu \int j_\rho(x',z')\, G(x,z,x',z')\, ds'$$

where $\varphi^{in}(x,z)$ is the incident field, which consists of lowest mode terms. The integration extends over the cross-sectional area of the block. Equation 4) may be interpreted as saying that the total field in the guide is equal to the sum of the contributions from all sources. The term φ^{in} represents the part due to the source at infinity; the integral is the part due to the sources in the dielectric. The field from one current filament in the dielectric would be $i\omega\mu G j_\rho$, hence the field from all the filaments is obtained by integration over the block.

The Green's function G has been found by Schwinger, and it is given as:

$$5)\ G = \frac{i}{a} \sum_{1}^{\infty} \frac{\sin\frac{n\pi x}{a} \sin\frac{n\pi x'}{a}}{K_n} e^{iK_n|z - z'|}$$

where $K_n = \sqrt{k^2 - (\frac{n\pi}{a})^2}$; $k = \frac{2\pi}{\lambda}$, where λ is the free space wavelength. For the assumed dimensions of the guide $K_1 = K$ is real, while all other K_n's are pure imaginary, thus: $K_n = i\sqrt{(\frac{n\pi}{a})^2 - k^2}$ $n > 1$. K_n is the propagation constant of the TE_{on} mode in the guide, and by assumption only the TE_{01} propagates. The Green's function is found by observing that G will be a solution of the wave equation with a unit line source singularity at $x = x'$, $z = z'$. It will have to vanish on the side walls $x = 0, a$, since it is the tangential component of the electric field there, and must behave like outgoing waves at infinity. G may be expanded in terms of the indicated modes, and the coefficients $\frac{1}{K_n}$ are found by making the z derivative of G have the discontinuity at $z = z'$ proper to a unit line source.

The polarization current may be found easily from Maxwell's equations as:

$$6)\ j_p(x,z) = -\left(\frac{\varepsilon}{\varepsilon_0} - 1\right) i\omega\varepsilon_0\, \varphi(x,z)\ .$$

Here ε is the complex dielectric constant of the block measured in rationalized M.K.S. units. We shall call $\varepsilon/\varepsilon_0$ just ε hereafter, whence Equation 4) becomes:

$$7)\ \varphi(x,z) = \varphi^{in}(x,z) + (\varepsilon - 1)\, k^2 \int \varphi(x'z')\, G(x,z,x',z')\, ds'$$

This is the fundamental integral equation of the problem. We now wish to use it to obtain the impedance elements.

4. The Even Case

Let us first consider the case where the field is an even function around the z-axis. This corresponds to open-circuiting the guide in the z=0 plane, or what is the same thing, to placing a "magnetic wall" at z = 0. The lowest mode incident field will be taken to be of unit amplitude, whence it is:

$$8)\ \varphi^{in}_e(x,z) = \sin\frac{\pi x}{a}\ \cos Kz = \Psi_e\ (x,z)$$

The Green's function must be modified so it satisfies the conditions of the magnetic wall, that is $\frac{\partial G}{\partial z} = 0$, since the electric field is a maximum of an open circuit; whence we take the even function:

$$9)\ G_e = G(x,z,x',z') + G(x,z,x',-z')$$

$$= \frac{i}{a} \sum_{1}^{\infty} \frac{\sin \frac{n\pi x}{a} \sin \frac{n\pi x'}{a}}{K_n} \left[e^{iK_n |z-z'|} + e^{iK_n |z+z'|} \right]$$

The total field in the guide is then:

$$10)\ \varphi_e = \psi_e(x,z) + (\varepsilon-1)\, k^2 \int G_e(x,z,x',z')\, \varphi_e\, (x',z')\, ds'$$

To find the lowest mode field, we select the term n = 1 of G_e, and obtain far to the left of the obstacle,

$$11)\ \varphi \text{lowest mode} = \sin \frac{\pi x}{a} \left[\cos Kz + (\varepsilon-1)\, k^2 \frac{2i}{Ka} e^{-iKz} \int \sin \frac{\pi x'}{a} \cos Kz' \varphi_e(x',z')ds' \right]$$

$$= \sin \frac{\pi x}{a} \left[\cos Kz + (\varepsilon-1)\, k^2 \frac{2i}{Ka} e^{-iKz} \int \psi_e(x',z')\, \varphi_e(x',z')\, ds' \right]$$

Let us therefore define the voltage as:

$$12)\ V_e(z) = \cos Kz + (\varepsilon-1)\, k^2 \frac{2i}{Ka} e^{-iKz} \int \psi_e(x',z')\, \varphi_e(x',z')\, ds'$$

since this is proportional to the amplitude of the electric field.

In the z = 0 plane the voltage reduces to:

$$13)\ V_e = 1 + (\varepsilon-1)\, k^2 \frac{2i}{Ka} \int \psi_e\, \varphi_e\, ds'$$

The current and voltage are related by transmission line equations, that is:

$$14)\ \frac{\partial V}{\partial z} = iKZ_o I$$

Z_o is the characteristic impedance of the guide and is arbitrary, since we may incorporate any multiplying factor into the definitions of voltage and current. Let us therefore take $Z_o = 1$, whence all impedances we may compute will be relative to the guide impedance. From Equations 12) and 14) the current is given by:

$$15)\qquad I_e(z) = i \sin Kz - (\varepsilon-1)\, k^2 \frac{2i}{Ka} e^{-iKz} \int \psi_e\, \varphi_e\, ds'$$

Then in the z = 0 plane:

$$16)\quad I_e = -(\varepsilon - 1)\, k^2 \frac{2i}{Ka} \int \psi_e \varphi_e \, ds'$$

The integral equation 10) may be rewritten by separating real and imaginary terms. The Green's function may be written as

$\frac{2i}{Ka} \sin \frac{\pi x}{a} \sin \frac{\pi x'}{a} \cos K_n z \cos K_n z' + G'_e$. The first term when placedin the integral gives

$$(\varepsilon - 1)\, k^2 \frac{2i}{Ka} \sin \frac{\pi x}{a} \cos K_n z \int \sin \frac{\pi x'}{a} \cos K_n z' \, \varphi_e (x', z') \, ds'$$

$$= (\varepsilon - 1)\, k^2 \frac{2i}{Ka} \psi_e \int \varphi_e \psi_e \, ds'$$

Combining this with the term ψ_e gives just $V_e \psi_e$. The integral equation becomes:

$$17)\quad \varphi_e = V_e \psi_e + (\varepsilon - 1)\, k^2 \int G'_e (x, z, x', z')\, \varphi_e (x', z')\, ds'$$

where the function G'_e is:

$$18)\quad G'_e = -\frac{1}{Ka} \sin \frac{\pi x}{a} \sin \frac{\pi x'}{a} \left[\sin K \left| z - z' \right| + \sin K \left| z + z' \right| \right]$$

$$+ \frac{i}{a} \sum_{2}^{\infty} \frac{\sin \frac{n\pi x}{a} \sin \frac{n\pi x'}{a}}{K_n} \left[e^{iK_n \left| z - z' \right|} + e^{iK_n \left| z + z' \right|} \right]$$

G'_e is pure real, since we have assumed that only the first mode propagates, so all the K_n's are pure imaginary.

Let us now define a new function $\varphi_e(x,z)$ by the relation

$$19)\quad \varphi_e(x,z) = -V_e \phi_e(x,z)$$

In terms of ϕ_e Equation 17) becomes:

$$20)\quad \psi_e(x,z) = -\phi_e(x,z) + (\varepsilon - 1)\, k^2 \int G'_e (x, z, x', z')\, \phi_e(x', z')\, ds'$$

while, on inserting 19) into 16), we obtain:

$$\frac{I_e}{V_e} = \frac{1}{Z_{11} + Z_{12}} = (\varepsilon - 1)\, k^2 \frac{2i}{Ka} \int \psi_e(x,z)\, \phi_e(x,z)\, ds$$

Equations 20) and 21) are the formal solution to the problem. Equation 20) is an integral equation from which the field Φ_e can be found, while 21) gives the impedance elements. Hence the value of $Z_{11} + Z_{12}$ can be obtained. To find Z_{11} and Z_{12} separately, we now turn to a consideration of what happens when the field is an odd function around z = 0. We may expect to obtain $Z_{11} - Z_{12}$ from this, by the analogy between Equations 2) and 3).

5. The Odd Case

An entirely analogous procedure can be carried through for the odd case. We now choose the incident electric field so it vanishes at z = 0, thus:

$$22)\quad \varphi_o^{in}(x,z) = \sin\frac{\pi x}{a}\sin Kz = \psi_o(x,z)$$

The Green's function must also be modified, so that it vanishes at z = 0. We then have:

$$23)\quad G_o = G(x,z,x',z') - G(x,z,x',-z')$$

$$= \frac{i}{a}\sum_{1}^{\infty}\frac{\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}}{K_n}\left[e^{iK_n|z-z'|} - e^{iK_n|z+z'|}\right]$$

The total field in the guide is then:

$$24)\quad \varphi_o(x,z) = \psi_o(x,z) + (\mathcal{E}-1)k^2\int G_o(x,z,x',z')\,\varphi_o(x',z')\,ds'$$

The voltage is found directly to be:

$$25)\quad V_o(z) = \sin Kz - (\mathcal{E}-1)k^2\,\frac{2}{Ka}\,e^{-iKz}\int\psi_o(x',z')\,\varphi_o(x',z')\,ds'$$

On the plane z = 0, we have

$$26)\quad V_o = -(\mathcal{E}-1)k^2\,\frac{2}{Ka}\int\psi_o(x',z')\,\varphi_o(x',z')\,ds'$$

The current is found from the transmission line equation to be:

$$27)\quad I_o(z) = -i\cos Kz + (\mathcal{E}-1)k^2\,\frac{2}{Ka}\,e^{-iKz}\int\psi_o(x',z')\,\varphi_o(x',z')\,ds'$$

On the plane z = 0, we have:

$$28)\quad I_o = -i + (\mathcal{E}-1)k^2\,\frac{2}{Ka}\int\psi_o(x',z')\,\varphi_o(x',z')\,ds'$$

Separating the imaginary term out of the integral equation 24), we find:

$$29)\quad \varphi_0(x,z) = \psi_0(x,z)\left[1 + (\varepsilon-1)\,k^2\,\frac{2i}{Ka}\int \psi_0(x',z')\,\varphi_0(x',z')\,ds'\right]$$

$$+\,(\varepsilon-1)\,k^2\int G_0'\,(x,z,x',z')\,\varphi_0\,(x',z')\,ds'$$

Here G_0' is found to be:

$$30)\quad G_0' = -\,\frac{1}{Ka}\,\sin\frac{\pi x}{a}\,\sin\frac{\pi x'}{a}\left[\sin K\,|z-z'| - \sin K\,|z+z'|\right]$$

$$+\,\frac{i}{a}\sum_{2}^{\infty}\frac{\sin\frac{n\pi x}{a}\,\sin\frac{n\pi x'}{a}}{K_n}\left[e^{iK_n|z-z'|} - e^{iK_n|z+z'|}\right]$$

Using the definition of current in Equation 28) gives:

$$31)\quad \varphi_0(x,z) = iI_0\,\psi_0(x,z) + (\varepsilon-1)\,k^2\int G_0'\,(x,z,x',z')\,\varphi_0(x',z')\,ds'$$

Defining a new function $\Phi_0(x,z)$ by the relation;

$$32)\qquad \psi_0(x,z) = -iI_0\,\Phi_0(x,z)$$

equations 31) and 26) become respectively:

$$33)\quad \psi_0(x,z) = -\,\Phi_0(x,z) + (\varepsilon-1)\,k^2\int G_0'(x,z,x',z')\,\Phi_0(x',z')\,ds'$$

$$34)\quad \frac{V_0}{I_0} = Z_{11} - Z_{12} = (\varepsilon-1)\,k^2\,\frac{2i}{Ka}\int \psi_0(x',z')\,\Phi_0(x',z')\,ds'$$

Equations 33) and 34) are the formal solutions to this part of the problem. Since all the details are entirely equivalent to those in the consideration of the even case, it has not been thought necessary to work out all the steps leading from 22) to 34).

6. Variational Formulation of the Impedance Elements.

While 20), 21), 33) and 34) constitute formal solutions to the problem, they do not of themselves provide any useful information. This is because it is not possible in general to solve the integral equations 20) and 33). Instead, various approximate methods must be used.

It is possible to rewrite the equations so that the approximate methods used will be self-correcting. Take Equation 20), multiply by $\phi_e(x,z)$, and integrate over the obstacle; then divide through by $\left[\int \phi_e \psi_e \, ds\right]^2$. This gives the modified form of the equation:

$$35)\quad \frac{1}{\int \phi_e \psi_e \, ds} = \frac{-\int \phi_e^2(x,z)\, ds + (\varepsilon - 1) k^2 \iint \phi_e(x,z) G_e'(x,z,x',z') \phi_e(x',z')\, ds' ds}{\left[\int \phi_e \psi_e \, ds\right]^2}$$

or, using Equation 21) to simplify the left side:

$$36)\quad (Z_{11}+Z_{12})(\varepsilon-1)k^2 \frac{2i}{Ka} = \frac{-\int \phi_e^2(x,z)\, ds + (\varepsilon-1) k^2 \iint \phi_e(x,z)\, G_e(x,z,x',z') \phi_e(x',z')\, ds' ds}{\left[\int \phi_e(x,z) \psi_e(x,z)\, ds\right]^2}$$

From the way in which the problem has been set up in terms of even and odd functions, the integrals extend over only half the cross-sectional area. The Green's function G_e was set up using an image theory, and it is clear that the sum of the even and odd Green's functions is twice the total Green's function (Equation 5). If we had taken the original integral equation (Equation 7), and separated it into even and odd parts, we would have obtained certain factors of 2, which are accounted for in going from 36) to 37). If we extend the integration over the entire cross-section, we may replace the even Green's function G_e' by G', the entire real Green's function, since the odd part integrates to zero. Taking care of the factors of 2 thereby introduced gives:

$$37)\quad (Z_{11} + Z_{12}) \frac{i(\varepsilon - 1) k^2}{Ka} =$$

$$- \frac{\int \phi_e^2(x,z)\, ds - \frac{1}{2}(\varepsilon-1) k^2 \int\int \phi_e(x,z) G'(x,z,x',z') \phi_e(x',z')\, ds ds'}{\left[\int \phi_e(x,z) \psi_e(x,z)\, ds\right]^2}$$

In Equation 37) the integrals extend over the entire cross-section of the dielectric. We have replaced G'_e by G' because the odd part gives no contribution to the integral. G' is given by:

$$38)\quad G' = -\frac{1}{Ka}\sin\frac{\pi x}{a}\sin\frac{\pi x'}{a}\sin K\,|z-z'| + \frac{i}{a}\sum_{2}^{\infty}\frac{\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}}{K_n}\,e^{iK_n|z-z'|}$$

We shall now show that the function which minimizes Equation 36) is the same function which satisfies the integral Equation 21). Suppose $\phi_e(x,z)$ is the minimizing function. Call the right hand side $f(\phi_e)$, and form $f(\phi_e + b\eta)$, where b is a parameter and η an arbitrary function. Differentiate with respect to b and set $b = 0$. From the assumption that ϕ_e is the minimizing function, it follows that $\frac{d}{db} f(\phi_e + b\eta)\Big|_{b=0} = 0$. Performing the differentiation and then setting $b = 0$ gives:

$$39)\quad 0 = \int\eta(x,z)\,ds\left[\begin{array}{l}\phi_e(x,z)\int\phi_e(x',z')\,\psi_e(x',z')\,ds' - \psi_e(x,z)\int\phi_e^2(x',z')ds' \\ -(\varepsilon-1)\,k^2\int\phi_e(x',z')G'_e(x,z,x',z')ds'\cdot\int\phi_e(x'',z'')\psi_e(x'',z'')\,ds'' \\ +(\varepsilon-1)k^2\psi_e(x,z)\iint\phi_e(x',z')G'_e(x',z',x'',z'')\,\phi_e(x'',z'')ds'ds''\end{array}\right]$$

Since $\eta(x,z)$ is an arbitrary function, it follows from a fundamental theorm in the calculus of variations that the expression in brackets is zero. This may be written in the form:

$$40)\quad \left[\phi_e(x,z) - (\varepsilon-1)\,k^2\int G'_e(x,z,x',z')\,\phi_e(x',z')ds'\right]\cdot\int\phi_e(x',z')\psi_e(x',z')ds'$$
$$-\,\psi_e(x,z)\int\phi_e(x',z')ds'\left[\phi_e(x',z')-(\varepsilon-1)k^2\int G'_e(x',z',x'',z'')\phi_e(x'',z'')ds''\right] = 0$$

If now ϕ_e satisfies the integral equation 21), the first term reduces to $\psi_e(x,z)\int\phi_e(x',z')\psi_e(x',z')ds'$ on substituting in the bracket. The second term becomes, upon substitution in the inner bracket, $-\psi_e(x,z)\int\phi_e(x',z')\psi_e(x',z')ds'$. Hence Equation 40) is satisfied by the solution of the integral Equation 21), and

conversely. It follows that the true field ϕ_e is the function which minimizes Equation 36).

Since 36) is minimized by the true field, we can find successive approximations to the true field by use of it. Suppose we assume a form for ϕ_e which contains several parameters. The integrations in 36) are then performed, and the result minimized with respect to the parameters. The process will usually be repidly convergent if a good trial function is chosen.

An entirely similar analysis can be carried through with Equations 33) and 34). Again an expression is obtained which is minimized by the true field. Approximations are used to obtain the impedance elements.

So far the analysis has been identical with that used by Schwinger in his treatment of the dielectric post. It now remains to choose good trial functions, and actually calculate the impedances.

7. Fields in an Infinite Block

In order to obtain useful results, we should obtain a reasonable approximation to the field in the dielectric. Some work has been done previously on the fields in a waveguide with an infinitely long dielectric strip. The modification in the cutoff wavelength for various modes was found by N.H. Frank [Rad.Lab.Rpts. T-9 and 43-1] and also by L. Pincherle [Phys. Rev. 66, p. 118]. Both correctly point out that the only types of fields which can be set up are those whose electric or magnetic vector, respectively, has no component normal to the dielectric interface. There is one set of modes corresponding to the TE_{n0} modes in ordinary waveguides for which the electric vector has only a y-component. Clearly these will be the only modes in the finite block generated when the incident field has only a y-component, as it does for the lowest mode fields.

For the fields of this infinite block let us assume:

$$E_y = A(x)\, e^{-jhz} \tag{41}$$

Here h is the propagation constant of the fields and will in general be complex. From Maxwell's equations we then obtain:

$$42)\quad H_x = -\frac{h}{\omega\mu} A(x)\, e^{-jhz}$$

$$H_z = \frac{j}{\omega\mu} A'(x)\, e^{-jhz}$$

We must have $A(x) = 0$ at $x = 0$ and $x = a$, and all components of field must be continuous at the dielectric interfaces, which requires $A(x)$ continuous, $A'(x)$ continuous. The requirement that E_y satisfies the wave equations fixes the form of $A(x)$ as a sine or cosine function of αx, where $\alpha^2 = k^2 - h^2$. Since $k^2 = \omega^2 \mu \varepsilon$ is different inside the dielectric from its value outside the dielectric α has different values inside from outside. The conditions at the two interfaces then yield a set of equations for the coefficients of the sine and cosine functions, and elimination gives an equation from which the values of α may be found. This equation is simple only in the special cases of a block directly against the wall, and a centered block. Writing α_1 for the value of α inside the dielectric, and α_2 for the value outside, we have the equation for a block against the wall:

$$43)\quad \alpha_2 \tan \alpha_1 d + \alpha_1 \tan \left[\alpha_2 (a - d)\right] = 0$$

and for a centered block:

$$44)\quad \alpha_1 \tan \alpha_1 \frac{d}{2} - \alpha_2 \cot \left[\alpha_2 \frac{a - d}{2}\right] = 0$$

α_1 and α_2 are connected by the relation:

$$45)\quad \alpha_1^2 - \alpha_2^2 = k_1^2 - k_2^2$$

$$= \left(\frac{2\pi}{\lambda}\right)^2 (\varepsilon - 1)$$

Equations 43) and 44) were solved numerically by Frank and Pincherle (loc.cit.), and graphs were given of the solutions.

Equations 43) and 44) can be solved approximately under the condition that d is small compared to a. Since the equations are transcendental, there will be an infinite set of roots, which correspond to the TE_{no} roots in the empty guide. The approximate solution for the first root of equation 43) is:

$$46)\quad \alpha_2 = \frac{\pi}{a}\left[1 - \frac{4}{3}\pi^2 \frac{d^3}{\lambda^2 a}(\varepsilon - 1)\right]$$

while for the first root of 44):

$$47)\quad \alpha_2 = \frac{\pi}{a}\left[1 - \frac{4ad}{\lambda^2}(\varepsilon - 1)\right]$$

In each case the second term in the brackets is assumed to be such that its square may be neglected. The values of the propagation constants may be found by direct substitution. Suppose that the dielectric is lossy, so that the dielectric constant has the real part ε', the imaginary part ε''. The attenuation may then be found. Under the same approximation, the attenuation in db/foot is given by:

$$48)\quad a)\quad A = 5450\, \varepsilon''\, \frac{1}{\lambda} \left(\frac{d}{a}\right)^3 \frac{1}{\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}}$$

$$b)\quad A = 1650\, \varepsilon''\, \frac{1}{\lambda}\, \frac{d}{a}\, \frac{1}{\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}}$$

in the two cases respectively. These are found by finding the propagation factor h from α, and then taking its imaginary part. A suitable conversion factor transforms this from nepers to db/foot. It is evident that the attenuation is much larger when the block is centered than when it is against the wall, since the ratio $\frac{d}{a}$ is small. For all other roots of 43) or 44) the attenuation will be much greater than for the lowest, since as far as they are concerned the waveguide is below cutoff.

8. The Finite Block Results

We may reasonably expect that if we represent the field in the finite block in terms of the modes of the infinite strip, the result will be rapidly convergent if the block is at all long. In fact, because of the large attenuation of the higher mode terms, the field will be given principally by a lowest mode term if the length of the block is comparable to a wavelength. Therefore, if we assume the entire field is given by a lowest mode term, the error can be expected to be small if the block is sufficiently long.

If the block is narrow enough, we may obtain a highly accurate result. From dimensional arguments, the second term in the numerator of the right side of Equation 37) is of the order $(\varepsilon-1)\frac{d}{a}$ times the first term. If the width is small, the functions Φ and G may be assumed constant over the width. The second term then involves two integrations over the width, the first term only one. The second term thus will be of the order d^2, the first term only of order d. The numerical factor is found by a more detailed consideration. If this factor is small, we may neglect

the second term, and obtain the approximate form:

$$49)\quad (Z_{11} + Z_{12})\,\frac{i(\varepsilon - 1)k^2}{ka} \cong -\,\frac{\int \phi_e^2 ds}{\left[\int \phi_e \psi_e \, ds\right]^2}$$

We may easily show that the function which minimizes the right-hand side is just $\phi_e = \psi_e$. To get this, we use the Schwartz inequality, which states that if f and g are any two functions:

$$50)\quad \left[\int fg ds\right]^2 \leq \int f^2 ds \cdot \int g^2 ds$$

or it may be rewritten:

$$51)\quad \frac{\int f^2 ds}{\left[\int fg ds\right]^2} \geq \frac{1}{\int g^2 ds}$$

Equality is obviously given by f = g, whence it follows that if g is given, the least value of the fraction on the left is obtained for f = g. Applying this to Equation 49), with $f = \phi_e$, $g = \psi_e$ gives:

$$52)\quad (Z_{11} + Z_{12})\,\frac{i(\varepsilon - 1)k^2}{Ka} \cong -\,\frac{1}{\int \psi_e^2 ds}$$

ψ_e is given by Equation 8) as $\psi_e = \sin\frac{\pi x}{a}\cos Kz$. Substituting this into 52) and evaluating the integral, keeping only first powers of $\frac{d}{a}$, gives;

$$53)\quad (Z_{11} + Z_{12})\,\frac{i(\varepsilon - 1)k^2}{Ka} = -\,\frac{2}{\lambda d}\,\frac{1}{1 + \frac{\sin K\ell}{K\ell}}\csc^2\frac{\pi x_0}{a}$$

Upon replacing i by -j we obtain finally:

$$54)\quad Z_{11} + Z_{12} = -j\,\frac{a}{d}\,\frac{1}{\varepsilon - 1}\left(\frac{\lambda}{\lambda_g}\right)^2 \frac{\lambda_g}{2\pi\ell}\,\frac{2}{1 + \frac{\sin\frac{2\pi\ell}{\lambda_g}}{\frac{2\pi\ell}{\lambda_g}}}\csc^2\frac{\pi x_0}{a}$$

We may apply the same analysis to the odd functions that we applied to the even. The approximate form is obtained:

$$55)\quad \frac{1}{Z_{11} - Z_{12}} \, \frac{i(\varepsilon-1)k^2}{Ka} \cong - \frac{\int \psi_0^2 \, ds}{\left[\int \phi_0 \psi_0 \, ds\right]^2}$$

As before, the right side is minimized for $\phi_0 = \psi_0 = \sin \frac{\pi x}{a} \sin Kz$. Upon evaluating the integral, we obtain:

$$56)\quad \frac{1}{Z_{11} - Z_{12}} \, \frac{i(\varepsilon-1)k^2}{Ka} \cong - \frac{2}{\ell d} \, \frac{1}{1 - \frac{\sin K\ell}{K\ell}} \csc^2 \frac{\pi x_0}{a}$$

and upon simplifying:

$$57)\quad Z_{11} - Z_{12} \cong j(\varepsilon-1) \frac{d}{a} \left(\frac{\lambda_2}{\lambda}\right)^2 \frac{\pi\ell}{\lambda_g} \left(1 - \frac{\sin \frac{2\pi\ell}{\lambda_g}}{\frac{2\pi\ell}{\lambda_g}}\right) \sin^2 \frac{\pi x_0}{a}$$

From these we can obtain the reflection coefficient R and the standing wave ratio η. These are given by the relations:

$$58)\quad R = \frac{(Z_{11} + Z_{12})(Z_{11} - Z_{12}) - 1}{(Z_{11} + Z_{12} + 1)(Z_{11} - Z_{12} + 1)}$$

$$59)\quad \eta = \left(\frac{1 + |R|}{1 - |R|}\right)^2$$

Since ε is complex, the impedances will be complex, and so will R. If we assume $(\varepsilon-1)\frac{d}{a}$ to be sufficiently small, we obtain the approximate formula:

$$60)\quad R \cong + j(\varepsilon-1) \frac{d}{a}\left(\frac{\lambda_g}{\lambda}\right)^2 \frac{2\pi\ell}{\lambda_g} \sin^2 \frac{\pi x_0}{a}$$

$$61)\quad \eta = \left[\frac{1 + |\varepsilon-1| \frac{d}{a}\left(\frac{\lambda_g}{\lambda}\right)^2 \frac{2\pi\ell}{\lambda_g} \sin^2 \frac{\pi x_0}{a}}{1 - |\varepsilon-1| \frac{d}{a}\left(\frac{\lambda_g}{\lambda}\right)^2 \frac{2\pi\ell}{\lambda_g} \sin^2 \frac{\pi x_0}{a}}\right]^2$$

In formula 61) the quantities d, a, x_0, and ℓ can all be measured directly. λ is known, λ_g can be computed, and η may be found by some simple measurements. Hence formula 61) may be used to determine the dielectric constant, or rather its magnitude. The phase of the dielectric constant may be found by determining the position of the maximum voltage, and using the formula:

$$62) \quad \mathrm{Im}\ \varepsilon = -|\varepsilon - 1| \cos \frac{4\pi z+}{\lambda_g}$$

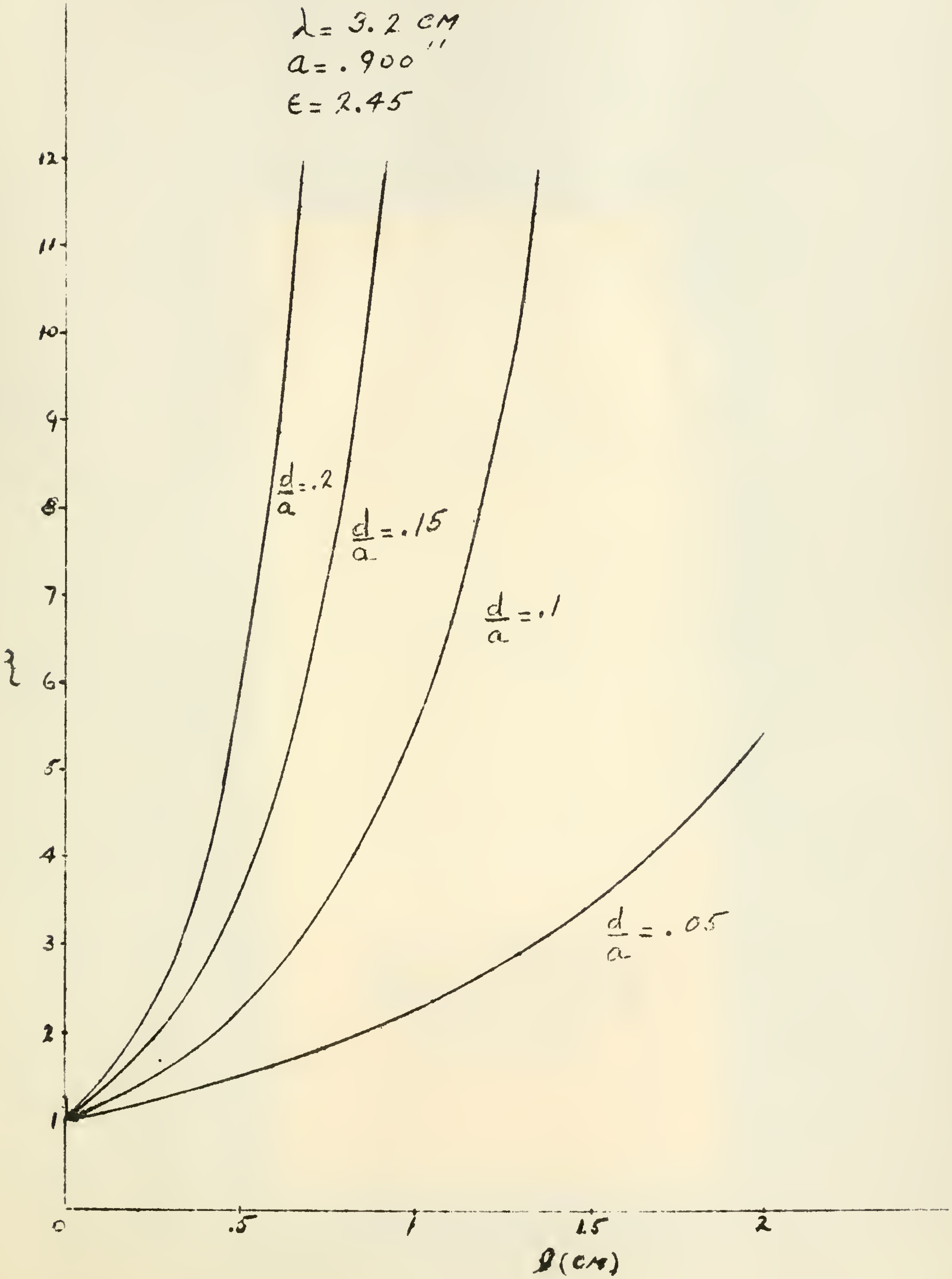
Here $z+$ denotes the position of the maximum far to the left of the block, measured from the center of the block. Formula 62) is only valid under the restricting condition $(\varepsilon - 1)\frac{d}{a} < < 1$.

9. Conclusion.

The next step would be to compute higher approximations to the field, using Equation 37) and the corresponding formula for the odd case. The next approximation to the field that we would use would be a term proportional to the lowest mode field that can be established in the dielectric. This is then substituted into Equation 37), and the integrals evaluated. The resulting expression is lengthy and cumbersome, and yields very little information beyond that given by the narrow block approximation. Also, it will probably not be accurate if the block is significantly wide, since in that case the field should be given by a superposition of modes. These will be strong near the left end of the block, and rapidly attenuate as we move to the right. The assumption of a single mode providing most of the field can be shown to be equivalent to saying that $(\varepsilon - 1)\frac{d}{a}$ is small, since the contribution of the next mode will be of that order of magnitude times the first. This amounts to using the narrow block theory, and if we actually evaluate the integrals, and then expand in powers of $(\varepsilon - 1)\frac{d}{a}$, the greatest contribution to the impedances can be shown to be given just by 54) and 57). This analysis shows that the narrow block theory is about as accurate an expression as can be obtained.

It has been pointed out by Schwinger that the dielectric post is a resonant structure. Equation 61) shows that the dielectric block also will be resonant, since it is possible for the denominator to vanish. This will occur for a particular length of block if the width and positions are given. If, keeping the frequency and thickness fixed, we vary the position of the block, the centered block has the shortest resonant length. For fixed thickness and position, the resonant length is to good approximation inversely proportional to the frequency. A plot of standing wave ratio against length for fixed thickness, position, and frequency is here given.

STANDING WAVE RATIO OF CENTERED BLOCK



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